J. Fluid Mech. (1993), *vol.* 250, *pp.* 635–650

Dynamics of liquid jets revisited

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(Received 21 May 1992 and in revised form 7 December 1992)

In this paper we investigate the long-wavelength approximations of the equations governing the motion of an inviscid liquid jet. Using a formal perturbation expansion it will be shown that the one-dimensional equations presented by Lee (1974) are inconsistent. The inconsistency arises from the fact that terms which have been retained in the boundary conditions should have been rejected in view of the approximations made in the momentum equations. With the correct equations a number of anomalies between Lee's model and other models are eliminated. An explicit periodic solution to the nonlinear evolution equations we have derived is presented. However, it turns out that the wavenumbers for which this solution is valid lie outside the range in which the long-wavelength approximations are applicable. In addition we present numerical solutions to the nonlinear equations we have derived. In the unstable regime we find that, as disturbances grow, the characteristic axial lengthscales of the major features are typically of the order of the radius of the jet. This casts some doubt on the validity of the long-wavelength approximations in the study of nonlinear liquid jet dynamics.

1. Introduction

Ever since the pioneering work of Rayleigh more than one hundred years ago, there has been a continuous interest in the dynamics of liquid jets and finite liquid columns. In the past decades this interest intensified due to the importance of liquid jets in various industrial applications (such as ink-jet printers) and the occurrence of liquid columns (in the form of a bridge between two solid end plates) in material science processes. The linear stability analysis of an inviscid jet of infinite length was presented by Rayleigh (1879). While that work provides insight into fundamental processes involved in the dynamics of jets, many aspects of importance such as the breaking process, drop formation and growth rate of disturbances are inherently nonlinear and as a result outside the reach of Rayleigh's analysis.

Attempts to understand the dynamics of a breaking jet have involved both experimental and theoretical studies. The experimental investigations range from studies of growth rates of disturbances and effects of viscosity by Donnelly & Glaberson (1966) and Goedde & Yuen (1970) to detailed investigations into the formation of drops by Pimbley & Lee (1977) and Vassallo & Ashgriz (1991). The theoretical investigations concentrate mainly on a perturbation expansion approach. Yuen (1968), Wang (1968), Nayfeh (1970) and Lafrance (1975) all used perturbation expansions to varying orders in the perturbation parameter to study finite-amplitude effects.

Given the fact that the problem of a breaking liquid jet is highly nonlinear involving large deformations of the free surface, it is natural to search for approximations of the governing equations which reduce the difficulty of the problem. The relative dimensions of the liquid jet naturally lead to so-called one-dimensional models to study jets and liquid columns. There are two fundamentally different approaches leading to a one-dimensional model. The first is a classical approach which, in essence, uses the 'shallow-water' approximations applied to the Euler equations in cylindrical coordinates. Basically one assumes that the radial momentum flux is small compared with the axial momentum flux which uncouples the radial momentum equation from the axial momentum equation. The second approach emanates from the theory that is based on the concept of a one-dimensional continuum, called a Cosserat continuum. Here one starts at the outset with a one-dimensional continuum and derives the equations and boundary conditions governing the deformations of the continuum using principles of invariance and constitutive assumptions. For details regarding this approach we refer to Green, Laws & Naghdi (1974*a*), Green, Naghdi & Wenner (1974*b*) and Green (1976).

Weber (1931) appears to have been the first who applied a shallow-water-type approximation to the equations governing the dynamics of a liquid jet. Based on the assumption that the pressure is constant in the radial direction, Weber derived equations which govern the motion of a slightly perturbed liquid jet. Using the same assumption Lee (1974) presents the equations governing the nonlinear behaviour of a jet. Subsequently Lee's model was employed in a number of studies concerning the dynamics of liquid jets (e.g. Pimbley 1976, Pimbley & Lee 1977, Torpey 1989). This model has also been used extensively in studies on the dynamics of finite liquid columns. We refer to, for example, Meseguer (1983) and Sanz (1985).

Neither Weber (1931) nor Lee (1974) provided a formal derivation of the equations they presented. Crucial is their *ad hoc* assumption of constant pressure in a radial cross-section of the jet. Although this assumption can be justified for a liquid jet, great care has to be taken regarding the exact nature of the approximation in the momentum equations in relation to the boundary conditions. Namely, in principle one should be able to derive the approximate equations by means of a formal perturbation expansion of the Euler equations with the ratio of the radius and the axial lengthscale as the perturbation parameter. Consistency would require all terms up to a specified order in the perturbation parameter to be retained in the momentum equations and the boundary conditions. However, it turns out that in the equations presented by Weber (1931) and Lee (1974) terms of a certain order were neglected in the radial momentum equation while higher-order terms in the boundary conditions were retained. The inconsistency becomes apparent when one compares the model of Lee and Weber with the Cosserat model in the linear (inviscid) limit. There is no reason to believe that these models should yield different results, as is the case, in this limit. Starting from a velocity potential formulation for the unsteady liquid jet, Moiseev (1965) did present a formal derivation of the approximate equations in the long-wavelength limit. However, his expression for the curvature is erroneous and no detailed discussion of results is given.

In §2 of this paper we show how the equations in the long-wavelength approximation can be derived by means of a formal perturbation expansion. In §3 we consider the linearized equations for the case of a liquid jet. It is shown that the consistent approximation of the governing equations brings the one-dimensional theory based on the Euler equations in line with the linear Cosserat theory. Periodic solutions of the nonlinear equations we have derived in §2 will be studied in §4. It will be shown how explicit analytical expressions for nonlinear long-wavelength disturbances on a liquid jet may be obtained. In §5 we show how numerical solutions to the equations derived in §2 can be obtained. The development of the jet in the unstable regime will be studied numerically in §6. Finally, we present our conclusions in §7.

2. Governing equations

Let us consider a liquid jet consisting of an inviscid, incompressible liquid. We assume that the jet is axisymmetric initially and that subsequent perturbations of the jet are axisymmetric so that there is no azimuthal dependence of the variables. The momentum equations are then given by

$$\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial r'}+w'\frac{\partial u'}{\partial z'}=-\frac{1}{\rho}\frac{\partial p'}{\partial r'}, \tag{1}$$

$$\frac{\partial w'}{\partial t'}+u'\frac{\partial w'}{\partial r'}+w'\frac{\partial w'}{\partial z'}=-\frac{1}{\rho}\frac{\partial p'}{\partial z'}, \tag{2}$$

in which primes indicate dimensional quantities, u' and w' denoting the radial and axial components of the velocity respectively, p' the pressure and ρ the density of the liquid. Note that body forces have been neglected. The incompressibility and irrotationality conditions yield

$$\frac{1}{r'}\frac{\partial}{\partial r'}(r'u')+\frac{\partial w'}{\partial z'}=0, \tag{3}$$

$$\frac{\partial w'}{\partial z'}-\frac{\partial u'}{\partial r'}=0. \tag{4}$$

On the capillary surface $r' = R+f'(z',t')$ we have the kinematic condition

$$\frac{\partial f'}{\partial t'}=u'-w'\frac{\partial f'}{\partial z'}, \tag{5}$$

in which $f'(z',t')$ denotes the deviation of the surface from the undisturbed cylindrical surface of radius R. The dynamic condition which relates the pressure to the curvature of the capillary interface is given by

$$p'=\frac{\sigma}{(1+(\partial f'/z')^2)^{\frac{1}{2}}}\left[\frac{1}{R+f'}-\frac{1}{1+(\partial f'/\partial z')^2}\frac{\partial^2 f'}{\partial z'^2}\right], \tag{6}$$

in which σ notes the coefficient of surface tension. Finally there are two symmetry conditions on the symmetry axis, namely

$$u'(r'=0,z',t')=\frac{\partial w'}{\partial r'}(r'=0,z',t')=0. \tag{7}$$

In order to derive the dimensionless equations to be approximated by means of a perturbation expansion we have to consider various characteristic scales of the dimensional quantities. Let R denote a typical radial lengthscale and let A be a characteristic amplitude of free surface deformations. We define $\epsilon = A/R$ and assume $\epsilon \ll 1$. A typical pressure scale is given by σ/R and a characteristic timescale is found by considering the dispersion relation for a liquid jet with radius R (see e.g. Lamb 1932), namely

$$\omega^2=\frac{\sigma}{\rho R^3}\frac{I_1(kR)}{I_0(kR)}kR[k^2R^2-1], \tag{8}$$

in which I_0, I_1 are modified Bessel functions of the first kind and k denotes the wavenumber of disturbances. Let $T_0 = (\rho R^3/\sigma)^{\frac{1}{2}}$ be some typical timescale for capillaryrelated effects, and define a dimensionless wavenumber via $\delta = kR$, with $\delta \ll 1$ corresponding to the long-wavelength limit. In this limit we obtain from (8)

$$|\omega|^2 \sim \frac{1}{T_0^2} k^2 R^2 + O(k^4 R^4).$$

It follows that a typical timescale for long-wavelength disturbances is given by $T = T_0/\delta$. Finally, a typical axial lengthscale is given by the inverse of the dimensional wavenumber, i.e. $k^{-1} = \delta/R$. The dimensional quantities can now be made dimensionless via

$$r' = Rr,$$
$$z' = (R/\delta)\, z, \quad t' = (T_0/\delta)\, t,$$
$$f'(z', t') = Af(z, t), \quad p'(r', z', t') = (\sigma/R)\, p(r, z, t),$$
$$u'(r', z', t') = (A/T)\, u(r, z, t) = \delta U_0\, u(r, z, t), \quad w'(r', z', t') = U_0\, w(r, z, t),$$

where $U_0 = A/T_0$. The axial velocity scale follows from continuity considerations. We point out that, although the scaling introduced here is special for the problem under consideration, a similar scaling is standard in the shallow-water theory; see, for example, Whitham (1974).

Let us first consider the irrotationality condition (4). Substituting for the dimensional quantities yields

$$\frac{\partial w}{\partial r} = \delta^2 \frac{\partial u}{\partial z}. \tag{9}$$

Since we have assumed δ to be small it follows that

$$w(r, z, t) = W(z, t) + O(\delta^2), \tag{10}$$

where we define $W(z, t) \equiv w(r = 0, z, t)$ for convenience. The continuity equation (3) reads

$$\frac{1}{r}\frac{\partial (ru)}{\partial r} = -\frac{\partial w}{\partial z} = -W_z + O(\delta^2),$$

on using (10). In the above and all subsequent equations a subscript denotes a partial derivative with respect to the particular subscript. Upon integrating the above equation we obtain

$$u(r, z, t) = -\tfrac{1}{2} r W_z + O(\delta^2). \tag{11}$$

We can now substitute (11) into (9) and integrate with respect to r to obtain

$$w(r, z, t) = W(z, t) - \tfrac{1}{4}\delta^2 r^2 W_{zz} + O(\delta^4), \tag{12}$$

which, on substituting into the continuity equation yields after integration

$$u(r, z, r) = \tfrac{1}{2} r W_z + \tfrac{1}{16}\delta^2 r^3 W_{zzz} + O(\delta^4). \tag{13}$$

The procedure outlined above can be repeated to yield $u(r, z, t)$ and $w(r, z, t)$ to any desired order in δ. Note that (12) and (13) satisfy the symmetry conditions (7).

Let us next consider the radial momentum equation (1). Substituting for the dimensional variables yields

$$\frac{\partial p}{\partial r} = -\epsilon\delta^2 \left(\frac{\partial u}{\partial t} + \epsilon u \frac{\partial u}{\partial r} + \epsilon w \frac{\partial u}{\partial z}\right).$$

Using (12) and (13) and retaining only the terms up to $O(\epsilon\delta^2)$ we can integrate the above equation to give

$$p(r, z, t) = \hat{p}(z, t) + \epsilon\delta^2 \tfrac{1}{4} r^2 W_{zt} + O(\epsilon^2\delta^2, \epsilon\delta^4), \tag{14}$$

in which $\hat{p}(z,t)$ is an integration constant. In order to determine the integration constant we consider the dynamic boundary condition (6). Upon substituting for the dimensional variables we obtain

$$p(r,z,t)|_{r=1+\epsilon f} = \frac{1}{1+\epsilon f}\frac{1}{(1+\epsilon^2\delta^2 f_z^2)^{\frac{1}{2}}} - \frac{\epsilon\delta^2 f_{zz}}{(1+\epsilon^2\delta^2 f_z^2)^{\frac{3}{2}}}. \tag{15}$$

Since terms of $O(\epsilon^2\delta^2)$ were neglected in the radial momentum equation, consistency requires the same to be done in the boundary conditions so that (15) reduces to

$$p(r,z,t)|_{r=1+\epsilon f} = \frac{1}{1+\epsilon f} - \epsilon\delta^2 f_{zz} + O(\epsilon^2\delta^2, \epsilon\delta^4). \tag{16}$$

The value of the integration constant in (14) follows from the boundary condition for the pressure as given by (16). Hence

$$p(r,z,t) = \frac{1}{1+\epsilon f} - \epsilon\delta^2 f_{zz} + \tfrac{1}{4}\epsilon\delta^2(r^2-1)\,W_{zt} + O(\epsilon^2\delta^2, \epsilon\delta^4). \tag{17}$$

At this stage it is important to compare our results with the results of Lee (1974). Lee took the pressure to be constant in the radial direction, which effectively means that terms of $O(\epsilon\delta^2)$ were neglected in the radial momentum equation. However, terms of $O(\epsilon\delta^2)$ and $O(\epsilon^2\delta^2)$ were retained in the dynamic boundary condition, namely all terms in (15) were retained. It is hard to see how this can be justified in view of the perturbation expansion presented here.

Let us next consider the axial momentum equation. Substituting for the dimensional variables yields

$$\epsilon\frac{\partial w}{\partial t} + \epsilon^2 u\frac{\partial w}{\partial r} + \epsilon^2 w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z}. \tag{18}$$

Using (12), (13) and (17) to eliminate $w(r,z,t)$, $u(r,z,t)$ and $p(r,z,t)$ respectively from the above equation, yields

$$\epsilon W_t + \epsilon^2 W W_z - \tfrac{1}{4}\epsilon\delta^2 W_{zzt} = \frac{\epsilon f_z}{(1+\epsilon f)^2} + \epsilon\delta^2 f_{zzz} + O(\epsilon^2\delta^2, \epsilon\delta^4). \tag{19}$$

Finally, from the kinematic boundary condition in dimensionless form, namely

$$\frac{\partial f}{\partial t} = u - \epsilon w\frac{\partial f}{\partial z},$$

together with (12) and (13) for $u(r,z,t)$ and $w(r,z,t)$ we obtain the continuity relation

$$\epsilon f_t = -\tfrac{1}{2}(1+\epsilon f)\,\epsilon W_z - \epsilon^2 W f_z + \tfrac{1}{16}\epsilon\delta^2 W_{zzz} + O(\epsilon^2\delta^2, \epsilon\delta^4). \tag{20}$$

Equations (19) and (20) constitute the equivalent of the Boussinesq equations governing long-wavelength disturbances on an axially symmetric liquid jet. It is convenient to rewrite (19) and (20) in terms of the mean axial velocity defined by

$$\bar{w}(r,z,t) = \frac{1}{\pi(1+\epsilon f)^2}\int_0^{1+\epsilon f} 2\pi w(r,z,t)\, r\,\mathrm{d}r.$$

Substituting for $w(r,z,t)$ as defined by (12) in the above expression yields

$$\bar{w}(z,t) = W(z,t) - \tfrac{1}{8}\delta^2 W_{zz} + O(\epsilon^2\delta^2, \epsilon\delta^4),$$

from which we obtain

$$W(z,t) = \bar{w}(z,t) + \tfrac{1}{8}\delta^2 \bar{w}_{zz} + O(\epsilon^2\delta^2, \epsilon\delta^4). \tag{21}$$

On substituting for $W(z,t)$ as given by (21) and using the notation $\mathscr{W}(z,t) = \epsilon\bar{w}(z,t)$ and $\mathscr{F}(z,t) = 1+\epsilon f(z,t)$ we find that (19) and (20) can be written like

$$\mathscr{W}_t + \mathscr{W}\mathscr{W}_z - \frac{\delta^2}{8}\mathscr{W}_{zzt} = -\mathscr{P}_z, \tag{22}$$

$$(\mathscr{F}^2)_t + (\mathscr{F}^2\mathscr{W})_z = 0. \tag{23}$$

The pressure term $\mathscr{P}(z,t)$ in (22) is given by

$$\mathscr{P}(z,t) = 3 - 3\mathscr{F} + \mathscr{F}^2 - \delta^2\mathscr{F}_{zz},$$

which is an approximation of the capillary pressure as given by (15) in which only terms up to order ϵ^2 and $\epsilon\delta^2$ are included. This approximation is consistent with all approximations made to derive (22) and (23). Namely, consistency of the perturbation expansion effectively requires $\delta^2 \sim \epsilon$ since the terms of order $\epsilon\delta^4$ and $\epsilon^2\delta^2$ were assumed to be of the same order of magnitude in our expansion. Equations (22), (23) are therefore consistent approximations of the Euler equations up to order ϵ^3.

Before we proceed with a more detailed discussion let us compare the above equations with the equations presented by Lee (1974). As expected, (23) remains unaltered since it describes the conservation of mass. The left-hand side of (22) differs from Lee's equation in that the latter does not include the $O(\delta^2)$ term. The right-hand side of (22) differs significantly from the equation presented by Lee because, as remarked before, Lee included all the terms in the dynamic boundary condition.

3. Linear analysis

The linear equations are obtained by neglecting all terms of order ϵ^2 in (22), (23). We obtain

$$\bar{w}_t - \tfrac{1}{8}\delta^2\bar{w}_{zzt} = f_z + \delta^2 f_{zzz}, \tag{24}$$

$$2f_t + \bar{w}_z = 0. \tag{25}$$

Let us assume wave-like disturbances with $\bar{w}(z,t)$ and $f(z,t)$ of the usual type $e^{i(k'z'+\omega' t')}$, or in dimensionless form $e^{i(z+\omega t/\delta)}$ using the time- and lengthscales defined in the previous section. Substituting for $\bar{w}$ and f we obtain the dispersion relation

$$\omega^2 = \frac{4\delta^2(\delta^2-1)}{8+\delta^2}. \tag{26}$$

We point out that dispersion relation (26) is identical to that obtained by Body (1978) using the linear Cosserat equations. The improved accuracy of this dispersion relation compared with that obtained by means of Lee's model is pointed out in Bogy's paper.

Retaining only terms of order δ^2 in (26) we obtain

$$\omega^2 = -\tfrac{1}{2}\delta^2 + O(\delta^4),$$

which is indeed the long-wavelength approximation of dispersion relation (8). We note that in this limit (which is formally the limit of Lee's 1974 approximation) all disturbances grow exponentially in time, the growth rate increasing monotonically with increasing wavenumber. The highly unstable behaviour of a jet emanating from a nozzle as predicted by Lee's model must be attributed to this fact. This notable shortcoming of Lee's model is remedied when a consistent approximation of the governing equations is made. Namely, the jet-nozzle problem in which we take (24) and (25) as the governing equations yields the linearized Cosserat problem which was shown by Body (1978) to have stability properties similar to those derived by Keller, Rubinow & Tu (1973).

Retaining all terms up to and including order δ^4 we find that (26) reduces to

$$\omega^2 = -\tfrac{1}{2}\delta^2(1-\tfrac{9}{8}\delta^2)+O(\delta^6). \tag{27}$$

A comparison with (8) shows that our analysis is consistent with the notion that the Boussinesq approximation has the effect of including the second term in the long-wavelength approximation of the dispersion relation. It is not necessary to proceed with a detailed discussion of the linear equations (24), (25) since they are equivalent to the linearized equations which follow from the Cosserat model. For a discussion of the Cosserat model and the discrepancies between this model and Lee's model we refer to the review paper by Bogy (1979).

4. Nonlinear periodic solutions

Let us next investigate periodic solutions to the problem defined by (22) and (23). To that end we consider solutions of the form $\bar{w} = \bar{w}(\xi)$, $f = f(\xi)$ where $\xi = z-ct$, c denoting the phase speed of the periodic disturbance. Equations (22) and (23) can then be written like

$$-c\bar{w}'+\tfrac{1}{2}\epsilon(\bar{w}^2)'+\tfrac{1}{8}\delta^2 c\bar{w}'''-f'+\epsilon(f^2)'-\delta^2 f''' = 0, \tag{28}$$

$$-c((1+\epsilon f)^2)'+((1+\epsilon f)^2\epsilon\bar{w})' = 0, \tag{29}$$

where the prime denotes a derivative with respect to ξ. Note that (29) may be integrated directly. Consistency of the resulting expression in the limit $\epsilon\rightarrow 0$ determines the integration constant, hence to second order in ϵ we obtain

$$\epsilon\bar{w}/c = 2\epsilon f-3\epsilon^2 f^2+O(\epsilon^3). \tag{30}$$

Equation (28) may be integrated once to yield

$$-c\bar{w}-f+\tfrac{1}{2}\epsilon\bar{w}^2+\epsilon f^2+\tfrac{1}{8}\delta^2 c\bar{w}''-\delta^2 f''+A = 0,$$

in which A is an integration constant. Using (30) to eliminate $\bar{w}$ and employing the identity $f'' = \tfrac{1}{2}\mathrm{d}(f'^2)/\mathrm{d}f$ one can show that the above equation reduces to

$$f'^2 = \mathscr{C}(f)+O(\epsilon,\delta^2), \tag{31}$$

in which $\mathscr{C}(f) = \alpha f^3-\beta f^2+Af+B, \quad \alpha = \dfrac{8\epsilon(1+5c^2)}{3\delta^2(4-c^2)}, \quad \beta = \dfrac{4(1+2c^2)}{\delta^2(4-c^2)}.$

We assume that $\mathscr{C}(f)$ has three real roots. The unknowns A, B in (31) are determined by specifying the height of the unperturbed free surface and the amplitude of the free surface displacements. By definition, the height of the unperturbed free surface is given by $f = 0$ while the amplitude of the disturbance is equal to $f = 1$. These choices define $A = -\alpha$ and $B = \beta$. Hence (31) becomes

$$f'^2 = \mathscr{C}(f) = (1-f^2)(\beta-\alpha f). \tag{32}$$

The general solution of (32) can be expressed readily in terms of Jacobian elliptic functions. With the substitution $f = \cos\phi$ we find (cf. Abramowitz & Stegun 1970),

$$f(\xi) = 2\,\mathrm{cd}^2(\tfrac{1}{2}(\alpha+\beta)^{\frac{1}{2}}\xi\,|\,m)-1, \quad m = \frac{2\alpha}{\alpha+\beta}. \tag{33}$$

A complete description of the periodic problem requires a relation between the parameters δ and ϵ and the phase speed c – essentially the dispersion relation which includes finite-amplitude effects. This dispersion relation follows from the definition of the wavelength. Namely, we know that $f(\xi)$ undergoes a periodic motion between the zeros $f = -1$ and $f = 1$ of $\mathscr{C}(f)$. Hence, the dimensionless wavelength, equal to 2π by virtue of the choice of lengthscales, is given by

$$2\pi = \frac{4}{(\alpha+\beta)^{\frac{1}{2}}} K(m), \tag{34}$$

in which $K(m)$ is the complete elliptic integral.

As a test of our analysis so far, let us consider the linear limit of (33) and (34). In the linear limit we take $\epsilon \to 0$ so that $\alpha \to 0$ and hence $m \to 0$. In this limit we have $\mathrm{cd}(u\,|\,m) \to \cos u$ and $K(m) \to \frac{1}{2}\pi$ so that equations (33) and (34) reduce to

$$f(\xi) = \cos \beta^{\frac{1}{2}}\xi, \quad \beta^{\frac{1}{2}} = 1. \tag{35}$$

Equations (35) show that in the linear limit we obtain the desired sinusoidal deflection of the free surface as studied in §3. The relation $\beta^{\frac{1}{2}} = 1$ is, in fact, an alternative formulation of the dispersion relation (26).

It is well-known that in the linear limit the wavenumber $\delta = 1$ marks the transitions from the regime in which stable periodic solutions exist (for $\delta > 1$) to the unstable regime (for $\delta < 1$). Nayfeh (1970) has shown that the stability boundary changes to $\delta = 1 + \frac{3}{4}\epsilon^2$ when finite-amplitude effects are incorporated. It is interesting to investigate what the stability boundary is as predicted by our model. We have shown that periodic solutions exist when $c^2 > 0$. In the linear limit the boundary between stable periodic solutions and unstable exponential growth corresponds to the value $c = 0$. It seems reasonable to assume that the value $c = 0$ also marks the stability boundary in the nonlinear case. Taking $c = 0$ we obtain $\alpha = \frac{2}{3}\epsilon$ and $\beta = 1$ so that $m = 4\epsilon/(2\epsilon+3)$. Employing the approximation

$$K(m) = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}m + \tfrac{9}{64}m^2) + O(m^3),$$

and (34), yields the critical wavenumber

$$\delta = 1 - \tfrac{1}{12}\epsilon^2.$$

This result is in disagreement, qualitatively and quantitively, with the stability limit derived by Nayfeh (1970). This should not be surprising since, as was pointed out by Nayfeh, the timescale near the stability limit is such that our scaling introduced in §2 ceases to be valid. In addition we note that near the stability boundary $\delta \sim O(1)$ whereas our analysis was based on the assumption that $\delta \ll 1$. The range of wavenumbers for which the foregoing analysis holds is therefore outside the range of the perturbation expansion.

Further evidence about the questionable validity of the periodic solutions derived in this section follows from the dispersion relation (34). Employing the approximation of $K(m)$ as stated above, one can show that (34) reduces to

$$\frac{\beta}{4\pi^2} = 1 + \gamma + O(\gamma^2), \quad \gamma = \frac{3\alpha^2}{8\beta^2} \ll 1. \tag{36}$$

The dispersion relation (26) can now be used to express γ in terms of δ. Solving (36) for c^2 by using the definition of β we obtain

$$c_\epsilon(\delta)^2 = c(\delta)^2\left(1 + \frac{\epsilon^2(7\delta^2-4)^2}{6\delta^2(\delta^2-1)(\delta^2+8)} + O(\epsilon^4)\right). \tag{37}$$

We observe that incorporating finite-amplitude effects leads to second-order variations in the dispersion relation, which is in agreement with the work by Wang (1968) and

Nayfeh (1970). However, the term containing ϵ^2 in (37) is positive for $\delta > 1$. This implies that the phase speed of the propagating waves increases with increasing amplitude, which disagrees with the findings of Wang and Nayfeh.

5. Nonlinear aperiodic solutions: numerical treatment

In the case where $c^2 < 0$, the parameter $\xi = z - ct$ introduced in the previous section becomes complex so that (32) becomes a complex equation. While the solution to this equation can be obtained readily in terms of elliptic functions, the physical significance of the solution is unclear. The nonlinear behaviour of a jet in the unstable regime is, however, a very interesting problem. The coupled set of nonlinear partial differential equations (22) and (23) will therefore be solved numerically in order to study the nonlinear long-wavelength behaviour of a jet.

We assume that the jet is ejected from an orifice at a certain speed and that we move at this speed with the jet. We assume, in addition, that a periodic, long-wavelength disturbance is generated in the liquid jet. In our frame of reference the disturbance will then appear as a stationary wave with a growing amplitude. Owing to the periodic nature of the disturbance, the boundary conditions are as follows:

$$\mathscr{W}(0,t) = \mathscr{W}(2\pi,t) = 0, \qquad \mathscr{F}_z(0,t) = \mathscr{F}_z(2\pi,t) = 0. \tag{38}$$

In addition we have to specify some initial conditions. We assume that the jet is initially cylindrical but the initial velocity is assumed to be some nonzero periodic function of z, namely

$$\mathscr{F}(z,0) = 1, \qquad \mathscr{W}(z,0) = \mathscr{W}_0(z). \tag{39}$$

Note that continuity implies that $\mathscr{F}_t(z,0) = -\frac{1}{2}\mathscr{W}_0'(z)$.

Equations (22) and (23) will be discretized using a finite-element approach. To that end we require the variational formulation of these equations, which reads as follows.

Find $\mathscr{W}, \mathscr{F} \in [0,2\pi]$ such that for all $\Phi, \Psi \in [0,2\pi]$ the following equations are satisfied:

$$\int_0^{2\pi} [\Phi\mathscr{W}_t + \Phi\mathscr{W}\mathscr{W}_z + \tfrac{1}{8}\delta^2\Phi_z\mathscr{W}_{zt} - \Phi_z\mathscr{P}]\,\mathrm{d}z = 0,$$

$$\int_0^{2\pi} [\Psi\mathscr{F}_t + \Psi\mathscr{W}\mathscr{F}_z + \tfrac{1}{2}\Psi\mathscr{F}\mathscr{W}_z]\,\mathrm{d}z = 0. \tag{40}$$

The test function Φ is chosen such that $\Phi(z)|_{z=0,\,2\pi} = 0$ in accordance with the boundary conditions for $\mathscr{W}$ and we choose Ψ such that $\mathrm{d}\Psi/\mathrm{d}z|_{z=0,\,2\pi} = 0$ in view of the boundary conditions imposed on $\mathscr{F}$.

A discrete set of equations is obtained by dividing the domain $[0,2\pi]$ up into elements and approximating $\mathscr{W}$ and $\mathscr{F}$ via $\mathscr{W} = \Sigma_j w_j(t)\Phi_j$, $\mathscr{F} = \Sigma_j f_j(t)\Psi_j$. Here, $\Phi_j(z)$ and $\Psi_j(z)$ denote the usual finite-element basis functions with small support. In order to determine which basis functions are to be used we note that $\mathscr{P}$ contains second-order derivatives so that the requirement $\Phi_z\mathscr{P} \in L_2[0,2\pi]$ indicates that basis functions with continuous first derivatives across element boundaries are required. To satisfy this requirement we employ the cubic Hermite interpolating functions as the basis functions Φ_j and Ψ_j: at each node both the function values $(\mathscr{F}, \mathscr{W})$ and its spatial derivatives $(\mathscr{F}_z, \mathscr{W}_z)$ are treated as unknowns, cf. Strang & Fix (1973). The discrete equivalent of (22) and (23) can now be written in matrix–vector form:

$$\left.\begin{aligned} (\boldsymbol{M} + \delta^2/8\boldsymbol{S})\,\dot{\boldsymbol{w}} + \boldsymbol{G}(\boldsymbol{w})\,\boldsymbol{w} - \boldsymbol{r}(\boldsymbol{f}) = 0, \\ \boldsymbol{M}\dot{\boldsymbol{f}} + \boldsymbol{G}(\boldsymbol{f})\,\boldsymbol{w} + \tfrac{1}{2}\boldsymbol{G}(\boldsymbol{w})\boldsymbol{f} = 0, \end{aligned}\right\} \tag{41}$$

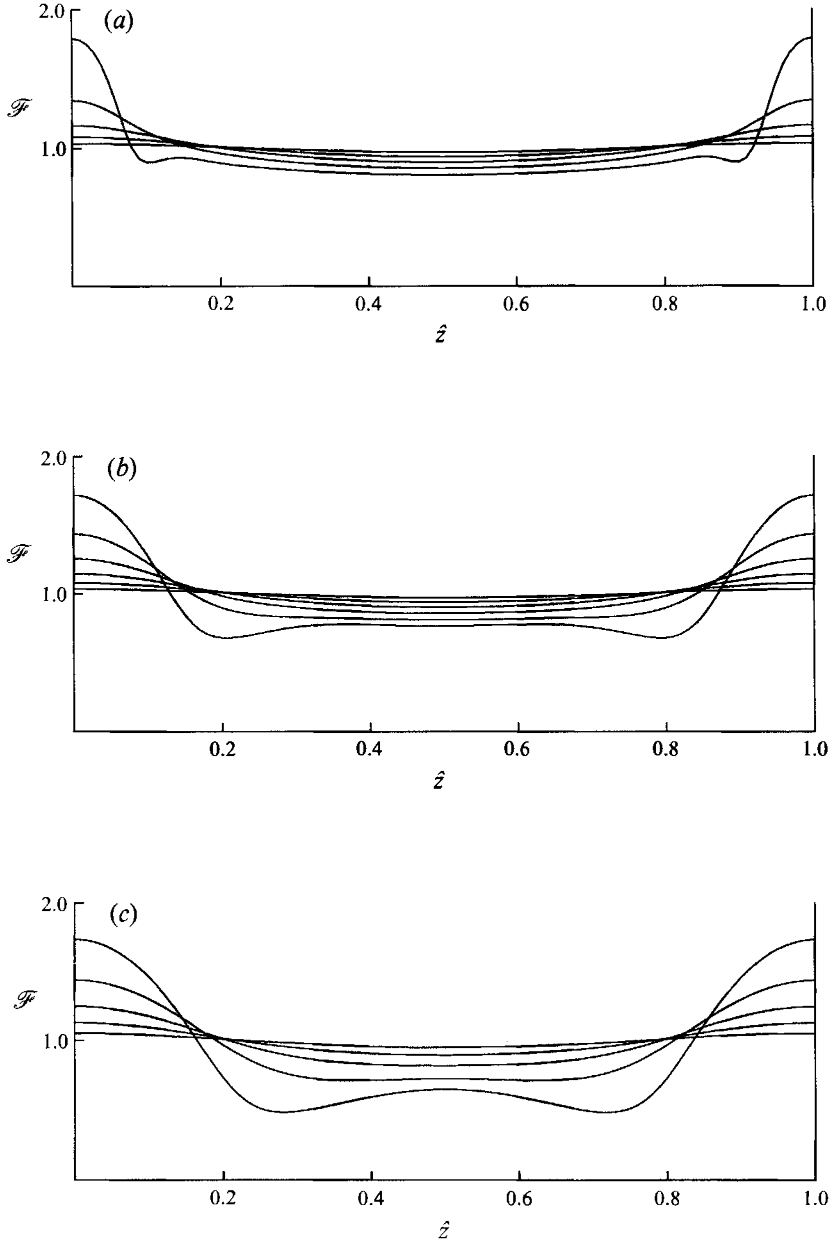

FIGURE 1 (*a–c*). For caption see facing page.

in which a dot denotes a time derivative and the components of the matrices are given by

$$M_{ij} = \int_0^{2\pi} \Phi_i \Phi_j \,\mathrm{d}z, \quad S_{ij} = \int_0^{2\pi} \frac{\mathrm{d}\Phi_i}{\mathrm{d}z}\frac{\mathrm{d}\Phi_j}{\mathrm{d}z}\,\mathrm{d}z,$$

$$G(x)_{ij} = \sum_k^{2N} x_k \int_0^{2\pi} \Phi_i \Phi_j \frac{\mathrm{d}\Phi_k}{\mathrm{d}z}\,\mathrm{d}z, \quad r_i = \int_0^{2\pi} \mathscr{P}\frac{\mathrm{d}\Phi_i}{\mathrm{d}z}\,\mathrm{d}z.$$

Note that we no longer distinguish between the basis functions Φ_j and Ψ_j. Without

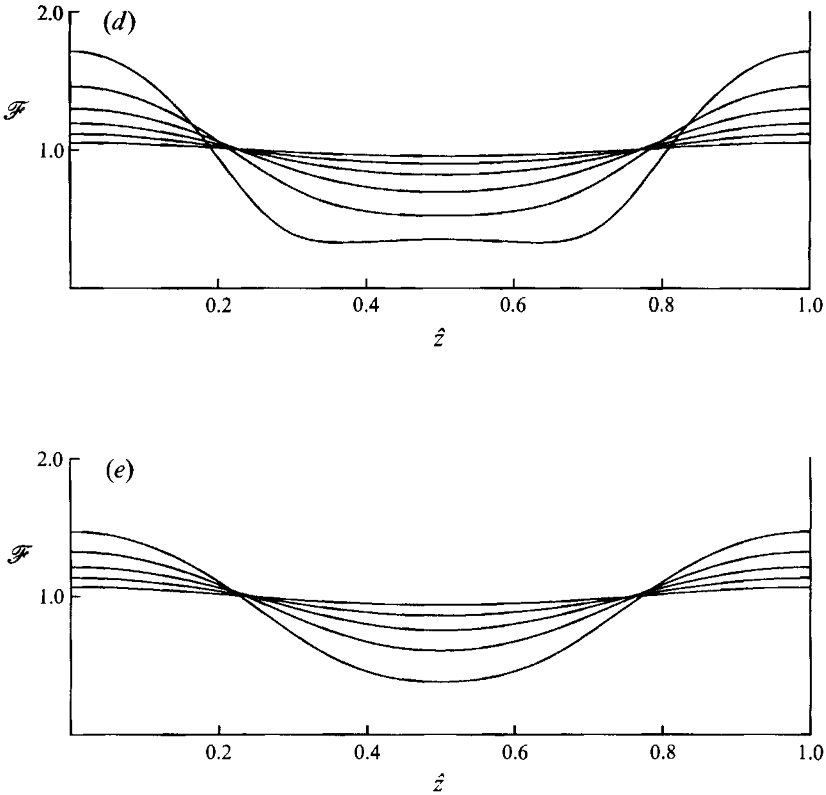

FIGURE 1. The evolution of the free surface of a jet for: (*a*) $\delta = 0.1$, $t = 0.1(0.1)0.5$; (*b*) $\delta = 0.3$, $t = 0.1(0.1)0.6$; (*c*) $\delta = 0.5$, $t = 0.15(0.15)0.75$; (*d*) $\delta = 0.7$, $t = 0.15(0.15)0.9$ and (*e*) $\delta = 0.9$, $t = 0.2(0.2)1.0$.

elaborating, it is important to note that the boundary conditions are periodic which leads to some changes in the matrices in (41).

The integration routine chosen to advance the solution of the initial-value problem (41) in time is a fourth-order Runge–Kutta scheme. Two systems of equations are to be solved repeatedly during the integration procedure. We point out that this can be done efficiently since the matrices $\boldsymbol{M}$ and $\boldsymbol{S}$ are constant throughout the integration procedure. Hence the LU-decomposition of these constant matrices needs to be calculated only once.

6. Results and discussion

All the numerical results presented in this section have been obtained by dividing the domain up into 100 elements of equal size. The time step in the Runge–Kutta scheme was taken to be $\Delta t = 0.00125$.

Let us consider the evolution of the free surface of the liquid jet for a given initial disturbance. We take the initial velocity distribution to be of the form

$$\mathscr{W}_0(z) = \epsilon_0 \sin(z)$$

with $\epsilon_0 = -0.1$. In figure 1 we show the free surface of the liquid jet for the wavenumbers $\delta = 0.1$, 0.3, 0.5, 0.7 and 0.9. For each wavenumber the free-surface

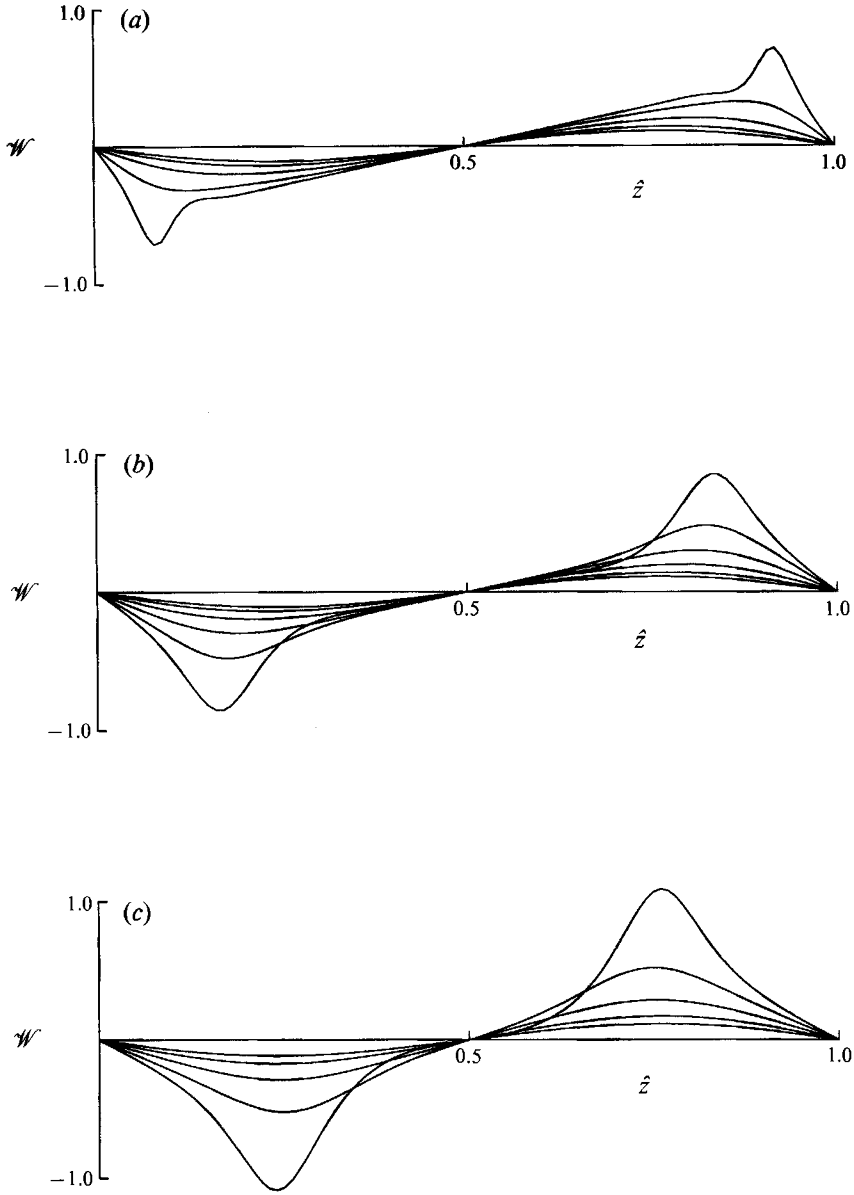

FIGURE 2(*a–c*). For caption see facing page.

shape is shown for a number of different times separated by equal time intervals. The corresponding axial velocity $\mathscr{W}(z,t)$ is shown in figure 2(*a–e*). In the plots the z-axis has been scaled with the wavelength via $\hat{z} = z/2\pi$.

Consider the case $\delta = 0.1$. We observe that, as time increases, the amplitudes of the swells at $\hat{z} = 0, 1$ increase with increasing rates. In contrast we observe that the amplitude at the point halfway between the swells does not increase significantly. This suggests that the neck will not form halfway between the swells. Indeed, if we allow the amplitudes to increase further we find that a neck starts to develop approximately one tenth of a wavelength away from the maximum amplitude (the neck forms near $\hat{z} = 0.1$ and $\hat{z} = 0.9$). Further evidence that the neck does not form halfway between the

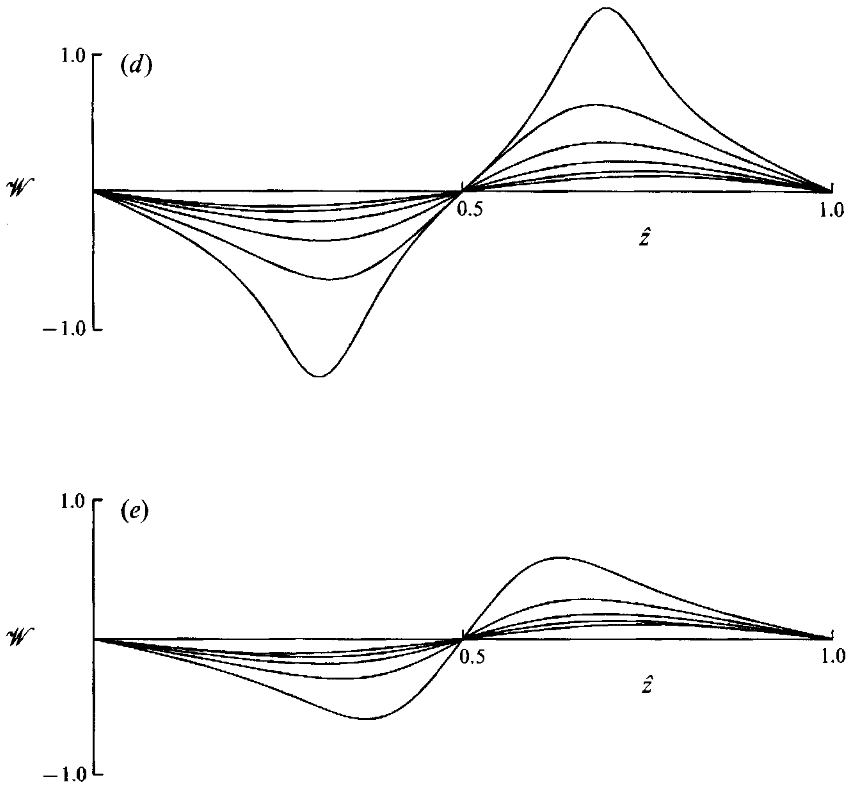

FIGURE 2. Plots of the axial velocity for: (*a*) $\delta = 0.1$, $t = 0.1(0.1)0.5$; (*b*) $\delta = 0.3$, $t = 0.1(0.1)0.6$; (*c*) $\delta = 0.5$, $t = 0.15(0.15)0.75$; (*d*) $\delta = 0.7$, $t = 0.15(0.15)0.9$ and (*c*) $\delta = 0.9$, $t = 0.2(0.2)1.0$.

wells is given by the axial velocity. Namely, a necessary requirement for necking to occur is a strongly divergent flow. The plot of the axial velocity for $\delta = 0.1$ (figure 2*a*) shows that at $\hat{z} = 0.5$ the flow is weakly divergent. As time increases we observe that a region with a strongly divergent flow starts to develop near $\hat{z} = 0.1$ and $\hat{z} = 0.9$.

For $\delta = 0.3$ (figures 1*b*, 2*b*) we observe a similar situation as for $\delta = 0.1$. However, the region of increasing amplitudes is somewhat wider for $\delta = 0.3$ than it was for $\delta = 0.1$. We also observe that a necking region starts to develop about one fifth of a wavelength away from the maximum amplitude. The plot of the axial velocity shows that this is indeed the region in which the flow becomes increasingly divergent for increasing time.

The cases $\delta = 0.5$ and $\delta = 0.7$ presented in figures 1(*c*), 2(*c*) and 1(*d*), 2(*d*) respectively show that as the wavenumber increases, the width of the swell increases while the region between the swells narrows. In addition we observe that the amplitudes in the necking region increase almost as fast as the amplitudes of the swells. For $\delta = 0.5$ we find that a neck develops a quarter of a wavelength away from the maximum amplitudes while for $\delta = 0.7$ the neck develops at about one third of a wavelength away from the maxima. We observe that for $\delta = 0.7$ the flow at the point $\hat{z} = 0.5$ becomes increasingly divergent for increasing time. Note that this is not the case for $\delta = 0.5$.

Finally consider the case $\delta = 0.9$ (figures 1*e*, 2*e*). We observe that the amplitude of the disturbance halfway between the swells increases faster than the amplitudes at the swells. Also observe that the flow becomes increasingly divergent near $\hat{z} = 0.5$ as time

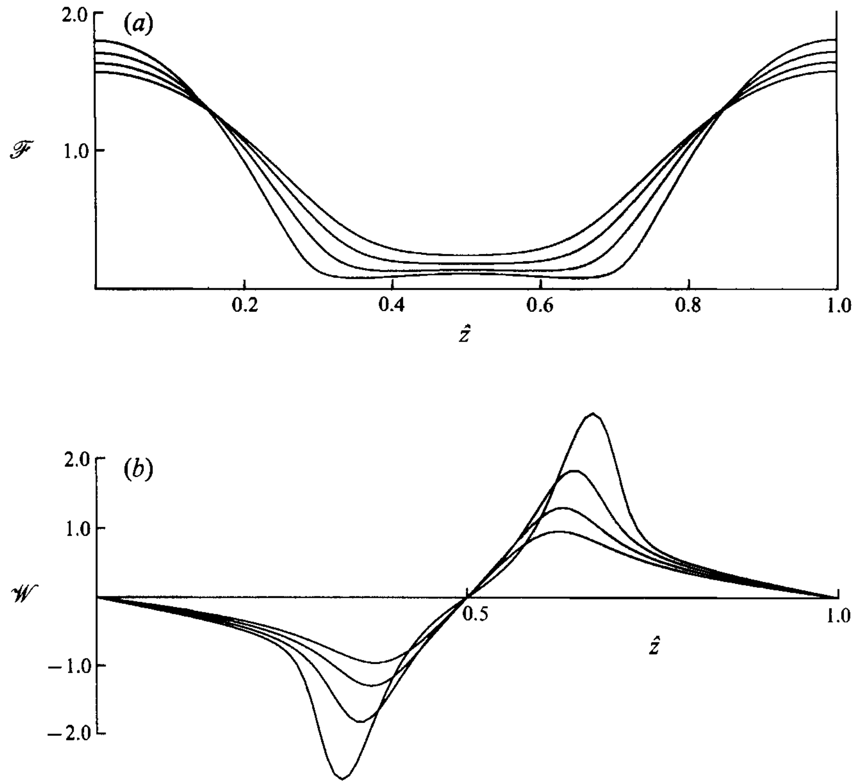

FIGURE 3. Evolution of the interface and the axial velocity for the case $\delta = 0.9$ at times $t = 1.1(0.05)\,1.25$.

increases, suggesting that the bifurcation point will be situated halfway between the swells.

The foregoing results are important in relation to the formation of satellite drops. We have seen that for $\delta \leqslant 0.7$ necking will not occur halfway between two swells but rather at some intermediate points. This implies that for that for $\delta \leqslant 0.7$ satellite droplets will always occur. We have also seen that, as δ decreases, a long flat region develops between the swells, the length of this region increasing for decreasing δ. This implies that the volume of the satellite drops increases with decreasing wavenumber δ. These results are in general agreement with those obtained by Lafrance (1975). Employing a perturbation expansion to third order and allowing the amplitudes to grow to one jet radius, he showed that satellite droplets are always present when the wavenumber of the disturbance is less than $\delta = 0.8$. A study by Shokoohi & Elrod (1987) of bifurcating liquid jets based on the numerical solution of the complete set of Navier–Stokes equations indicates, however, that satellite droplets will occur for any wavenumber $\delta < 1$. It is interesting that our one-dimensional model also predicts the formation of satellite droplets for $\delta > 0.8$. In figure 3 we show the evolution of the capillary interface close to the bifurcation point for the case $\delta = 0.9$. We note that while the necking region has a well-defined minimum for $t = 1.1$ the necking region becomes flat as time increases and assumes the characteristic shape observed for the smaller wavenumbers.

Some caution is required when we consider numerical results in which the amplitude of the disturbance is close to the initial radius of the jet. Given that in the approximations leading to (22) and (23) we have neglected terms of order ϵ^3, it is clear that allowing the amplitudes to increase much beyond half the initial radius of the jet is likely to introduce significant errors. At least as important is the inherent limitation of any one-dimensional model in which radial momentum effects are only partly included. Close to the bifurcation point of the jet the surface tension force is inversely proportional to the radius of the neck. This implies that prior to the bifurcation the necking region experiences a large inward acceleration during which radial momentum effects become increasingly important. Including only first-order radial momentum effects as we have done, is a severe approximation in that case.

Further limitations of the one-dimensional model become apparent when we consider figures 1 and 2. Namely, the important assumption that the typical axial lengthscale is of order R/δ does not hold as the amplitudes of the disturbances increase. For example, the calculations for the long-wavelength disturbance in which $\delta = 0.1$ (figures 1*a*, 2*a*) show that as the amplitudes increase, the typical lengthscale of the major features is one tenth of a wavelength. This implies that for large amplitudes the characteristic axial scale is given by the radius R rather than R/δ.

7. Conclusions

In this paper we have derived the equations governing the motion of axisymmetric long-wavelength disturbances on a liquid jet. It is shown that with a consistent perturbation approximation the one-dimensional model presented by Lee (1974) is modified and brought into line with the Cosserat model presented by Green (1976).

Exact periodic solutions can be obtained for the equations we have derived. However, we find that these solutions do not agree with results obtained by others. This discrepancy is due to the fact that periodic solutions only exist for a range of wavenumbers for which the long-wavelength approximation is not applicable.

Numerical solutions of the nonlinear equations are presented for the case of long-wavelength disturbances which are known to be unstable. A classification of the evolution of long-wavelength disturbances on a jet is possible. Disturbances with a wavenumber less than $\delta = 0.7$ typically show a rapid increase in the amplitudes of the swells while the region between the swells remains fairly flat until close to the bifurcation point. As the bifurcation point is approached, two necks start to develop at the ends of the flat region adjacent to the swells. The length of the flat region between the swells increases with increasing wavelength. Necks and swells of disturbances with a wavenumber in the range $0.7 < \delta < 1$ grow at more or less equal rates: there is no flat region between the swells. Only when the bifurcation point is approached do we find that growth rates of the swells and the necks start to differ significantly. The neck region becomes flat and two separate necks start to develop. A consequence of this behaviour is that satellite droplets will always be formed. The volume of the satellite drop increases with increasing wavelength.

The applicability of the one-dimensional approach we have discussed in this paper is limited by a number of different factors. First of all there is the inherent limitation of the long-wavelength approximation in describing the dynamics of disturbances with a wavelength less than the circumference of the jet. However, even for long waves the long-wavelength approximation fails when the amplitudes of disturbances has grown close to the initial radius of the jet. This failure is due to the increasing importance of the radial momentum effects when the radius of the neck decreases. In addition we find that as the wavelength of disturbances increases, the disturbances become increasingly localized for growing amplitudes. In fact, for large amplitudes the typical axial scale of the main features is of the order of the radius of the jet. Hence, the crucial assumption that the axial lengthscale is of the order of the radius of the jet divided by the wavenumber no longer holds when amplitudes have increased significantly.

The author is indebted to one of the referees for pointing out errors in the original manuscript. This research was financed by the Commission of European Communities under contract number B/SC1-900617.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1970 *Handbook of Mathematical Functions*, 9th edn. Dover.

BOGY, D. B. 1978 Use of one-dimensional Cosserat theory to study instability in a viscous liquid jet. *Phys. Fluids* **21**, 190–197.

BOGY, D. B. 1979 Drop formation in a circular liquid jet. *Ann. Rev. Fluid Mech.* **11**, 207–228.

DONNELLY, R. J. & GLABERSON, W. 1966 Experiments on the capillary instability of a liquid jet. *Proc. R. Soc. Lond.* A **209**, 547–556.

GOEDDE, E. F. & YUEN, M. C. 1970 Experiments on liquid jet instability. *J. Fluid Mech.* **40**, 495–511.

GREEN, A. E. 1976 On the non-linear behaviour of liquid jets. *Intl J. Engng. Sci.* **14**, 49–63.

GREEN, A. E., LAWS, N. & NAGHDI, P. M. 1974*a* On the theory of water waves. *Proc. R. Soc. Lond.* A **338**, 43–55.

GREEN, A. E., NAGHDI, P. M. & WENNER, M. L. 1974*b* On the theory of rods II. Developments by direct approach. *Proc. R. Soc. Lond.* A **337**, 485–507.

KELLER, J. B., RUBINOW, S. L. & TU, Y. O. 1973 Spatial instability of a jet. *Phys. Fluids* **16**, 2052–2055.

LAMB, H. 1932 *Hydrodynamics*, 6th edn. Cambridge University Press.

LAFRANCE, P. 1975 Nonlinear break up of a laminar liquid jet. *Phys. Fluids* **18**, 428–432.

LEE, H. C. 1974 Drop formation in a liquid jet. *IBM J. Res. Dev.* **18**, 364–369.

MESEGUER, J. 1983 The breaking of axisymmetric slender liquid bridges. *J. Fluid Mech.* **130**, 123–151.

MOISEEV, N. N. 1965 On some properties of fluid flow under the action of surface tension forces. *Prikl. Mat. Mek.* **29**, 1015–1022. (English translation: 1966 *Appl. Math. Mech.* **29**, 1197–1205.)

NAYFEH, A. H. 1970 Nonlinear stability of a liquid jet. *Phys. Fluids* **13**, 841–847.

PIMBLEY, W. T. 1976 Drop formation from a liquid jet: a linear one-dimensional analysis considered as a boundary value problem. *IBM J. Res. Dev.* **20**, 148–156.

PIMBLEY, W. T. & LEE, H. C. 1977 Satellite droplet formation in a liquid jet. *IBM J. Res. Dev.* **21**, 21–30.

RAYLEIGH, LORD 1879 On the instability of jets. *Proc. Lond. Math. Soc.* **10**, 4–13.

SANZ, A. 1985 The influence of the outer bath in the dynamics of axisymmetric liquid bridges. *J. Fluid Mech.* **156**, 101–140.

SHOKOOHI, F. & ELROD, H. G. 1987 Numerical investigation of the disintegration of liquid jets. *J. Comput. Phys.* **71**, 324–342.

STRANG, G. & FIX, G. J. 1973 *An Analysis of the Finite-Element Method.* Prentice-Hall.

TORPEY, P. A. 1989 A nonlinear theory for describing the propagation of disturbances on a capillary jet. *Phys. Fluids* A **1**, 661–671.

VASSALLO, P. & ASHGRIZ, N. 1991 Satellite formation and merging in liquid jet break up. *Proc. R. Soc. Lond.* A **433**, 269–286.

WANG, D. P. 1968 Finite amplitude effect on the stability of a jet of circular cross-section. *J. Fluid Mech.* **34**, 299–313.

WEBER C. 1931 Zum Zerfall eines Flüssigkeitsstrahles. *Z. Angew. Math. Mech.* **11**, 136–141.

WHITHAM, G. B. 1974 *Linear and Nonlinear Waves.* Wiley-Interscience.

YUEN, M. C. 1968 Non-linear capillary instability of a liquid jet. *J. Fluid Mech.* **33**, 151–163.